

A few remarks about linear operators and disconnected open sets in the plane

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Let E be a nonempty compact set with empty interior in the complex plane, and let $U \subseteq \mathbf{C}$ be a bounded open set such that $U \cap E = \emptyset$ and $\partial U = E$. For instance, U might be the union of all of the bounded components of $\mathbf{C} \setminus E$. We are especially interested here in the situation where E is connected and U has infinitely many connected components, and moreover where each neighborhood of each element of E contains infinitely many components of U . Of course, U can have only countably many components, since the plane is separable. Sierpinski gaskets and carpets are basic examples of such fractal sets E .

One can choose different orientations for the different components of U . This can be represented by a locally constant function on U with values ± 1 , where $+1$ corresponds to the standard orientation on \mathbf{C} . Alternatively, a locally constant function on $\mathbf{C} \setminus E$ with values 0 and ± 1 can be used to specify which complementary components are included in U as well as their orientations. Bergman and Hardy spaces on U can then be defined using holomorphic or conjugate-holomorphic functions on the components of U , depending on the choice of orientation of the component.

The topological activity in E indicated by its complementary components can also be described in terms of homotopy classes of continuous mappings from E into $\mathbf{C} \setminus \{0\}$. This is reflected in Fredholm indices of associated Toeplitz operators too. It is natural to allow different orientations on different components of U , in order to get different combinations of indices, which may involve many variations. At the same time, the standard orientation on all of U has some special features.

Perhaps the first point is that there are a lot of holomorphic functions in the usual sense on neighborhoods of \overline{U} , which are in particular very regular functions on \overline{U} that are holomorphic on U . Conversely, removable singularity results imply that a sufficiently well-behaved function on \overline{U} which is holomorphic on U is holomorphic on the interior of \overline{U} . For instance, this holds for continuously-differentiable functions on \overline{U} because E has empty interior, and for Lipschitz functions of order 1 on \overline{U} when E has Lebesgue measure 0. Note that the differential of a continuously-differentiable function f on \overline{U} which is holomorphic on $U_1 \subseteq U$ and conjugate-holomorphic on $U_2 \subseteq U$ vanishes on $\overline{U_1} \cap \overline{U_2}$. Since E is connected, $\overline{U_1} \cap \overline{U_2} \neq \emptyset$ when $U_1, U_2 \neq \emptyset$ and $U_1 \cup U_2 = U$.

Continuous functions that are holomorphic on both sides of a nice curve are holomorphic across the curve. Holomorphic functions that are real-valued on a curve yield continuous functions that are holomorphic on one side of the curve and conjugate-holomorphic on the other side, by taking the complex conjugate on one side. This is related to the reflection principle.

If E is sufficiently big, then there are a lot of continuous functions on \mathbf{C} that are holomorphic on $\mathbf{C} \setminus E$, given by Cauchy integrals of suitable measures on E . It is not as easy to work with the differential operator equal to $\bar{\partial}$ on the components of U with the standard orientation and to ∂ on the components of U with the opposite orientation when the orientations are variable.

Suppose that f is a continuously-differentiable complex-valued function on \bar{U} . If E is not asymptotically flat at a point $p \in E$, then the differential of f at p is uniquely determined by the restriction of f to E . Complex-linearity of the differential of f at p is then a significant condition, and not simply a question of choosing an extension of f on E so that the differential is complex-linear, as in the case of a smooth curve. If $\bar{\partial}f = 0$ on E , then there is more regularity for some commutators and Hankel-type operators corresponding to f and the standard orientation on all of U . There are also a lot of functions that satisfy this condition, which is equivalent to saying that $\bar{\partial}f(p) \rightarrow 0$ when $p \in U$ approaches the boundary. The analogous condition for variable orientations would ask that $\bar{\partial}f(p) \rightarrow 0$ or $\partial f(p) \rightarrow 0$ as $p \in U$ approaches the boundary through parts of U with the standard or opposite orientation. This implies that the differential of f is 0 at any $p \in E$ in the boundary of both parts of U , as before.

Even with variable orientations on the components of U , one can make use of Cauchy, conjugate-Cauchy, and other kernels on the individual components. Some regularity of functions on or around E can still be very helpful, but not in quite the same way as when the orientations are constant.

Note that there is normally no uniform orientation like the one inherited from the complex plane for fractal sets with topological dimension 1 bounding quasi-fractal sets with topological dimension 2 in \mathbf{R}^n when $n > 2$.

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